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MULTIPLE MIXING FROM WEAK HYPERBOLICITY BY THE HOPF ARGUMENT

YVES COUDÈNE, BORIS HASSELBLATT AND SERGE TROUBETZKOY

ABSTRACT. We show that using only weak hyperbolicity (no smoothness, compactness or exponential rates) the Hopf argument produces multiple mixing in an elementary way. While this recovers classical results with far simpler proofs, the point is the broader applicability implied by the weak hypotheses. Some of the results can also be viewed as establishing "mixing implies multiple mixing" outside the classical hyperbolic context.

1. INTRODUCTION

The origins of hyperbolic dynamical systems are connected with the efforts by Boltzmann and Maxwell to lay a foundation under statistical mechanics. In today's terms their fundamental postulate was that the mechanical system defined by molecules in a container is ergodic, and the difficulties of establishing this led to the search for *any* mechanical systems with this property. The motion of a single free particle (also known as the geodesic flow) in a negatively curved space emerged as the first and for a long time sole class of examples with this property. Even here, establishing ergodicity was subtle enough that initially this was only done for constantly curved surfaces by using the underlying algebraic structure. Eberhard Hopf was the first to go beyond this context, and his argument remains the main tool for deriving ergodicity from hyperbolicity in the absence of an algebraic structure (the alternative tool being the theory of equilibrium states). Our purpose is to show how much more than ergodicity it can produce. Specifically, in its original form the Hopf argument establishes ergodicity when the contracting and expanding partitions of a dynamical system are jointly ergodic. We present a recent refinement originally due to Babillot that directly obtains mixing from joint ergodicity of these two partitions. Further, we publicize the observation that the argument produces multiple mixing if the stable partition is ergodic by itself, and we give a simple proof of ergodicity of the stable foliation. Taken together, this gives a simple,

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self-contained general proof of multiple mixing of which Corollary 5.2 is a prototype.

Here is how the results in this paper can be applied together. Use the Hopf argument (Section 3) to establish mixing (or just total ergodicity), deduce that the stable partition is ergodic (Section 5), then apply the one-sided Hopf argument (Section 2) to obtain multiple mixing. We remark that our proofs are self-contained, quite short and do not use compactness, differentiability, or exponential behavior; nor are the W^i assumed to consist of manifolds. The step from ergodicity to multiple mixing does not need the full force of the usual notions of local product structure and absolute continuity. Indeed, in our applications to billiards (Theorem 5.5) and partially hyperbolic dynamical systems (Theorem 4.4), more information is available than needed for our results.

Hyperbolic dynamical systems on compact spaces enjoy even stronger stochastic properties, such as the Kolmogorov property and being measurably isomorphic to a Bernoulli system [4, Theorem 4.1]. Our purpose is to show how much follows from just the Hopf argument.

We conclude this introduction with the Hopf argument for ergodicity. Consider a metric space X with a Borel probability measure μ and a μ-preserving transformation $f\colon X \to X$. The stable partition of f is defined by

$$W^{ss}(x) := \{y \in X \mid d(f^n(x), f^n(y)) \xrightarrow[n\to+\infty]{} 0\} \tag{1}$$

DEFINITION 1.1. $\varphi\colon X \to \mathbb{R}$ is *subordinate* to W^{ss} or W^{ss}*-saturated* if there is a set $G \subset X$ with $\mu(G) = 1$ such that $x, y \in G$ and $y \in W^{ss}(x)$ imply $\varphi(x) = \varphi(y)$.

REMARK 1.2. In this case $\varphi^s(x) := \begin{cases} 0 & \text{if } W^{ss}(x) \cap G = \varnothing \\ \varphi(y) & \text{if } y \in G \cap W^{ss}(x) \end{cases} \overset{\text{a.e.}}{=} \varphi$ is (everywhere!) constant on stable sets.

THEOREM 1.3 (Hopf Argument)**.** *If (X,μ) is a metric Borel probability space, $f\colon X \to X$ μ-preserving, then any f-invariant $\varphi \in L^p(\mu)$ is W^{ss}-saturated.*

Proof. The Luzin Theorem gives $F_k \subset X$ with $\mu(X \smallsetminus F_k) < 2^{-k}$ and $\varphi \restriction F_k$ uniformly continuous. If $E_k := \left\{x \in X \mid \frac{1}{2} < \tau_{F_k} := \lim\limits_{N\to\infty} \frac{1}{N}\#\{0 \le n < N \mid f^n(x) \in F_k\}\right\}$, then, using the Birkhoff ergodic theorem,

$$\mu(X \smallsetminus E_k) = 2\int_{X\smallsetminus E_k} 1/2 \le 2\int_{X\smallsetminus E_k} 1-\tau_{F_k} \le 2\int 1-\tau_{F_k} = 2\int \chi_{X\smallsetminus F_k} = 2\mu(X\smallsetminus F_k) < 2^{1-k},$$

while for $x, y \in E_k$ there are $n_i \xrightarrow[i\to\infty]{} \infty$ with $\{f^{n_i}(x), f^{n_i}(y)\} \subset F_k$, since each has density $> 1/2$. If furthermore $y \in W^{ss}(x)$ and φ is f-invariant, then $\varphi(x) - \varphi(y) = \varphi(f^{n_i}(x)) - \varphi(f^{n_i}(y)) \xrightarrow[i\to\infty]{} 0$, which proves the claim on $\bigcup_{n\in\mathbb{N}}\bigcap_{k\ge n} E_k \overset{\text{a.e.}}{=} X$. $\square$

If f is invertible, then we define

$$W^{su}(x) := \{y \in X \mid d(f^{-n}(x), f^{-n}(y)) \xrightarrow[n\to+\infty]{} 0\},$$

to get

THEOREM 1.4. *If (X,μ) is a metric Borel probability space, $f\colon X\to X$ invertible μ-preserving, then any f-invariant $\varphi\in L^2(\mu)$ is W^{ss}- and W^{su}-saturated.*

DEFINITION 1.5. Let $f\colon X\to X$ be a Borel-measurable map of a metric space X. An f-invariant Borel probability measure μ is said to be ergodic (or f to be *ergodic* with respect to μ) if every f-invariant measurable set is either a null set or the complement of one. Equivalently, every bounded measurable f-invariant function φ is constant a.e.: $\varphi\circ f=\varphi\Rightarrow\varphi\overset{\text{a.e.}}{=}\text{const}$.

By analogy and as the link between Theorem 1.4 and ergodicity we define

DEFINITION 1.6. W^{ss}, W^{su} are said to be *jointly ergodic* if

$$\varphi\in L^2(\mu),\ W^{ss}\text{-saturated and }W^{su}\text{-saturated}\Rightarrow\varphi\overset{\text{a.e.}}{=}\text{const}.$$

THEOREM 1.7. *If (X,μ) is a metric Borel probability space, $f\colon X\to X$ invertible μ-preserving, W^{ss},W^{su}* jointly ergodic*, then f is ergodic.*

Although this paper gives a substantial strengthening of this classical conclusion, we note a well-known simple one that is not often made explicit. Since joint ergodicity is unaffected if we replace f by f^n, we actually have

THEOREM 1.8. *If (X,μ) is a metric Borel probability space, $f\colon X\to X$ invertible μ-preserving, W^{ss},W^{su}* jointly ergodic*, then f is totally ergodic.*

Here

DEFINITION 1.9. f is said to be *totally ergodic* if f^n is ergodic for all $n\in\mathbb{N}^*$.

REMARK 1.10. This is equivalent to having no roots of unity in the spectrum of the associated Koopman operator on L^2 and to having no adding machine or permutation on a finite set as a factor [16, p. 119]. We can, of course, conclude in Theorem 1.8 that f^n is ergodic for $n\in\mathbb{Z}\smallsetminus\{0\}$.

We can now state more explicitly the objectives of this paper.

- Explain how joint ergodicity of the partitions implies more than total ergodicity of f, namely mixing (Theorem 3.3).
- Give a nontrivial application (to partially hyperbolic dynamical systems, Theorem 4.4).
- Explain how the stronger assumption of ergodicity of W^{ss} (alone) implies even more, namely multiple mixing (Theorem 2.2).
- Establish criteria for ergodicity of W^{ss} (Theorem 5.1).
- Give nontrivial applications (e.g., to billiards, Theorem 5.5).

The approach that improves ergodicity to mixing and multiple mixing gives rise to a question which we make explicit here by way of previewing the approach.

For establishing ergodicity, there is the trivial step from Theorem 1.3 to Theorem 1.4 (f-invariant functions are f^{-1}-invariant). For establishing mixing, this is echoed below in the nontrivial step from Proposition 2.1 for $N=1$ to Theorem 3.1 (weak accumulation points of $\varphi\circ f^n$ are W^{ss}- *and* W^{su}-saturated) which is originally due to Babillot. We have no corresponding step for establishing multiple mixing from joint ergodicity of W^{ss} and W^{su}, that is, we do not know how to go from Proposition 2.1 for $N > 1$ to a corresponding statement about W^{ss}- *and* W^{su}-saturation.

PROBLEM. If X is a metric space, $f\colon X \to X$, μ an f-invariant Borel probability measure, $\varphi_i \in L^2(\mu)$, then is any weak accumulation point of $\prod_{i=1}^N \varphi_i \circ f^{\sum_{j=1}^i n_j}$ with $n_i \xrightarrow[n\to\infty]{} \infty$ W^{ss}-saturated *and* W^{su}-saturated?

An affirmative answer would say that in our results that conclude "mixing" one does, in fact, have multiple mixing.

1.1. Examples.

EXAMPLE 1.11. Transformations of the form $f \times (-1)\colon X \times \{1,-1\} \to X \times \{1,-1\}$, $(x, y) \mapsto (f(x), -y)$ are not mixing regardless of the ergodic properties of f. While in this case the finitary reduction given by the return map to $X \times \{1\}$ may produce a mixing transformation, the corresponding counterpart for flows is a suspension, in which the absence of mixing is deemed substantial.

While the dynamical systems in which we are interested are differentiable—either diffeomorphisms or flows—our interest is in the ergodicity and related properties of Borel probability measures invariant under the dynamical system. In the mechanical (that is, Hamiltonian) case, this would, for instance be the so-called Liouville volume. We mentioned geodesic flows as the original motivating examples, and we now add others to our discussion. For all of these we will prove multiple mixing via the Hopf argument, that is, without recourse to sophisticated results from entropy theory and the theory of measurable partitions in the context of hyperbolic dynamical systems.

EXAMPLE 1.12. The action of $\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$ on $\mathbb{R}^2$ projects to an area-preserving diffeomorphism $F_{\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)}\colon \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{T}^2$. Distances on lines parallel to the eigenline $y = \dfrac{\sqrt{5}-1}{2}x$ for the eigenvalue $\lambda_1 = \dfrac{3+\sqrt{5}}{2} > 1$ are expanded by a factor λ_1. Similarly, the lines $y = \dfrac{-\sqrt{5}-1}{2}x + \text{const.}$ contract by $\lambda_1^{-1} = \lambda_2 = \dfrac{3-\sqrt{5}}{2} < 1$.

EXAMPLE 1.13. More generally, any $A \in GL(m,\mathbb{Z})$ induces an automorphism F_A of $\mathbb{T}^m$ that preserves Lebesgue measure. We say that it is hyperbolic if A has no eigenvalues on the unit circle.

EXAMPLE 1.14 ([24, p. 104],[23, p. 49])**.** Likewise, $W := \left(\begin{smallmatrix}0&0&0&-1\\1&0&0&8\\0&1&0&-6\\0&0&1&8\end{smallmatrix}\right)$ induces a volume-preserving automorphism F_W of $\mathbb{T}^4$. The eigenvalues $2-\sqrt{3} \pm i\sqrt{4\sqrt{3}-6}$ lie on the unit circle and the eigenvalues $\lambda_\pm = 2+\sqrt{3} \pm \sqrt{2(3+2\sqrt{3})} \in \mathbb{R}$ satisfy $0 < \lambda_- < 1 < \lambda_+$. F_W is thus *partially hyperbolic*. The components of the eigenvectors

$$v^\pm := (-2-\sqrt{3} \pm \sqrt{2(3+2\sqrt{3})}, 3 \mp 2\sqrt{2(-3+2\sqrt{3})}, -6+\sqrt{3} \pm \sqrt{2(3+2\sqrt{3})}, 1)$$

are independent over $\mathbb{Q}$, i.e., generate a 4-dimensional vector space over $\mathbb{Q}$.

EXAMPLE 1.15 ([10, p. 67])**.** A billiard $\mathcal{D} \subsetneq \mathbb{T}^2$ is said to be *dispersing* if it is defined by reflection in the boundary of smooth strictly convex "scatterers."[1] If it has no corners or cusps, then Sinai's Fundamental Theorem of the theory of dispersing billiards [8, 21], see also [10, Theorem 5.70], establishes hyperbolic behavior of the billiard map.

EXAMPLE 1.16. Sinai's Fundamental Theorem also applies to *polygonal billiards with pockets*. These are noncircular billiards obtained from a convex polygon as follows: for each vertex add a disk whose interior contains this vertex and none other [11, Theorem 4.1].

EXAMPLE 1.17. The *Katok map* is a totally ergodic area-preserving deformation of $F_{\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)}$ that is on the boundary of the set of Anosov diffeomorphisms (hence not uniformly hyperbolic) and whose stable and unstable partitions are homeomorphic to those of $F_{\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)}$ [3, §1.3], [2, §6.3], [17, §2.2], [19].

1.2. **Ergodicity and related notions.** Since the *time-averages* or *Birkhoff averages* $\frac{1}{n}\sum_{i=0}^{n-1}\varphi\circ f^i$ converge a.e. (Birkhoff Pointwise Ergodic Theorem) and in L^2 (von Neumann Mean Ergodic Theorem), ergodicity is equivalent to time averages coinciding with space averages ($\int\varphi$); this conclusion was the actual object of the Maxwell–Boltzmann Ergodic Hypothesis. The motivation is that such functions φ represent *observables* by associating to each state of the system (each point in the domain of the dynamical system) a number that might be the result of an experimental measurement. We note that in this context we can use all L^p spaces ($p\in[1,\infty]$) interchangeably: for any $p\in[1,\infty]$ ergodicity of f is equivalent to f-invariant L^p functions being constant.

A simple nontrivial example of an ergodic transformation is $x\mapsto x+\alpha \pmod 1$ on $S^1=\mathbb{R}/\mathbb{Z}$ for irrational α (Kronecker–Weyl Equidistribution Theorem [18, Proposition 4.2.1]). The preceding examples are also ergodic (with respect to the area measure), but unlike an irrational circle rotation, they have stronger stochastic properties, and the aim of this note is to show that the Hopf argument yields them.

A colloquial motivation for these is that if φ represents the sugar concentration in a cup with a lump of sugar, then rotation of the cup does little to mix (and dissolve) the sugar.

DEFINITION 1.18. An f-invariant Borel probability measure is said to be *mixing* if two observables become asymptotically independent or uncorrelated when viewed as random variables:

$$\int\varphi\circ f^n\,\psi \xrightarrow[n\to\infty]{} \int\varphi\int\psi \quad \text{for all} \quad \varphi,\psi\in L^2. \tag{2}$$

[1]One can allow corners at considerable expense of additional effort [10, p. 69].

Equivalently,

$$\varphi \circ f^n \xrightarrow[n\to\infty]{\text{weakly}} \text{const.} \quad \text{for all} \quad \varphi \in L^2. \tag{3}$$

With test function $\psi \equiv 1$ in (2), the left-hand side is independent of n, which shows that the constant on the right-hand side of (3) is $\int \varphi$.

DEFINITION 1.19. μ is said to be *multiply mixing* if it is N*-mixing* for all $N \in \mathbb{N}$: For $\varphi_1, \dots, \varphi_N \in L^\infty$ and any L^2-weak neighborhood U of (the constant function) $\prod_{i=1}^N \int \varphi_i \, d\mu$ there is a $K \in \mathbb{R}$ such that $\prod_{i=1}^N \varphi_i \circ f^{\sum_{j=1}^i n_j} \in U$ whenever $n_i \geq K$ for $1 \leq i \leq N$. In short,

$$\prod_{i=1}^N \varphi_i \circ f^{\sum_{j=1}^i n_j} \xrightarrow[n_i\to\infty]{L^2\text{-weakly}} \prod_{i=1}^N \int \varphi_i \, d\mu \quad \text{for} \quad \varphi_i \in L^\infty.$$

Made explicit with test function φ_0, this means that $N+1$ observables become asymptotically independent as the time gaps between them go to infinity. Here, the left-hand side is parametrized by $\mathbb{Z}^N$, and the assertion can be checked by considering sequences $\psi_k = \prod_{i=1}^N \varphi_i \circ f^{\sum_{j=1}^i n_j(k)}$ with $n_i(k) \xrightarrow[k\to\infty]{} \infty$ and $\psi_k \xrightarrow[k\to\infty]{\text{weakly}} \psi$; then ψ is an accumulation point, and we describe these as "weak accumulation points $\psi_k \xrightarrow[k\to\infty]{\text{weakly}} \psi$ of $\prod_{i=1}^N \varphi_i \circ f^{\sum_{j=1}^i n_j(k)}$ with $n_i(k) \xrightarrow[k\to\infty]{} \infty$" or as "weak accumulation points of $\prod_{i=1}^N \varphi_i \circ f^{\sum_{j=1}^i n_j}$ as $n_i \to \infty$." N-mixing means that for $\varphi_i \in L^\infty$ there is only one weak accumulation point of $\prod_{i=1}^N \varphi_i \circ f^{\sum_{j=1}^i n_j}$ with $n_i \to \infty$, and it is $\prod_{i=1}^N \int \varphi_i \, d\mu$.

PROPOSITION 1.20. *An f-invariant Borel probability measure μ is N-mixing iff given any $\varphi_i \in L^2(\mu)$, any weak accumulation point of $\prod_{i=1}^N \varphi_i \circ f^{\sum_{j=1}^i n_j}$ with $n_i \to \infty$ is constant.*

Proof. "Only if" is clear. To prove "if", we recursively determine the constant.

First, take $\varphi_i \equiv 1$ for $i \neq 1$, including taking the test function $\varphi_0 \equiv 1$. Then the weak-accumulation statement becomes

$$\int \varphi_1 = \int \varphi_1 \circ f^{n_1} \cdot 1 \to \text{const.} \int 1 = \text{const.},$$

so the constant is $\int \varphi_1$ for each such subsequence, and thus $\varphi_1 \circ f^{n_1} \xrightarrow[n_1\to\infty]{\text{weakly}} \int \varphi_1$. By symmetry, $\varphi_i \circ f^{n_i} \xrightarrow[n_i\to\infty]{\text{weakly}} \int \varphi_i$ for all i. Next, if $\varphi_i \equiv 1$ for $i \notin \{1,2\}$, then

$$\int \varphi_1 \circ f^{n_1} \cdot \varphi_2 \circ f^{n_1+n_2} \cdot 1 = \int \varphi_2 \circ f^{n_2} \cdot \varphi_1 \xrightarrow[n_2\to\infty]{} \int \varphi_1 \int \varphi_2$$

by the first step, so

$$\varphi_1 \circ f^{n_1} \cdot \varphi_2 \circ f^{n_1+n_2} \xrightarrow[n_1,n_2\to\infty]{\text{weakly}} \int \varphi_1 \int \varphi_2$$

with like statements for any pair of the φ_i. This can be continued, and the existence of an accumulation point (by the Banach–Alaoglu Theorem) completes the proof. ☐

2. THE ONE-SIDED HOPF ARGUMENT YIELDS MULTIPLE MIXING

We note that the following uses no compactness or exponential contraction.

PROPOSITION 2.1 ([12, §3.3])**.** *If (X,μ) is a metric Borel probability space, $f\colon X \to X$ μ-preserving, $\varphi_i \in L^2(\mu)$, then weak accumulation points of $\prod_{i=1}^N \varphi_i \circ f^{\sum_{j=1}^i n_j}$ with $n_i \xrightarrow[n\to\infty]{} \infty$ are W^{ss}-saturated.*

Proposition 1.20 gives a strong immediate consequence of Proposition 2.1:

THEOREM 2.2. *f is multiply mixing if every W^{ss}-saturated $\varphi \in L^2$ is constant a.e.*

Proof of Proposition 2.1. By the Banach–Saks Lemma $\psi_n \xrightarrow[n\to\infty]{L^2\text{-weakly}} \psi$ has a subsequence for which $\dfrac{1}{n}\sum\limits_{k=0}^{n-1}\psi_{n_k} \xrightarrow[n\to\infty]{L^2} \psi$. Furthermore, $\psi_n \xrightarrow[n\to\infty]{L^2} \psi$ implies that there is a subsequence with $\psi_{n_k} \xrightarrow[k\to\infty]{\text{a.e.}} \psi$. This gives subsequences m_l, n_{i_k} with

$$\Psi_l := \frac{1}{m_l}\sum_{k=0}^{m_l-1}\psi_{n_{i_k}} \xrightarrow[l\to\infty]{\text{a.e.}} \psi.$$

Pointwise convergence makes this W^{ss}-saturated for bounded uniformly continuous functions: $p_{ij}^l := \varphi_i(f^{(n_i)_l}(x_j))$ for $j = 1,2$ with $x_2 \in W^{ss}(x_1)$ gives

$$\prod_{i=1}^N p_{i2}^l - \prod_{i=1}^N p_{i1}^l = \sum_{\ell=1}^N \Big[\prod_{i=1}^{\ell-1} p_{i2}^l\Big]\big[p_{\ell 2}^l - p_{\ell 1}^l\big]\Big[\prod_{i=\ell+1}^N p_{i1}^l\Big] \xrightarrow[l\to\infty]{} 0.$$

Approximate $\varphi_i^0 \in L^\infty \cap L^2$ within $1/k$ by bounded uniformly continuous φ_i^k and let $p_{ij}^l := \varphi_i^j \circ f^{(n_i)_l}$. Then weak limits (of subsequences if necessary) satisfy

$$\|\psi-\psi^k\| \le \varliminf_{l\to\infty} \Big\|\prod_{i=1}^N p_{ik}^l - \prod_{i=1}^N p_{i0}^l\Big\| \le \sum_{\ell=1}^N \prod_{i=1}^{\ell-1}\|p_{i2}^l\|_\infty \|p_{\ell 2}^l - p_{\ell 1}^l\|_2 \prod_{i=\ell+1}^N \|p_{i1}^l\|_\infty \xrightarrow[k\to\infty]{} 0$$

so, after passing to a subsequence, $\psi^k \xrightarrow{\text{a.e.}} \psi$, which is hence W^{ss}-saturated. ☐

In Example 1.12 the contracting lines have irrational slope, so the intersections of each with the circle $S^1 \times \{0\} \subset S^1 \times S^1 = \mathbb{T}^2$ are the orbit of an irrational rotation—whose ergodicity implies that the stable partition W^{ss} is ergodic [18, Proposition 4.2.2]. The "one-sided" Theorem 2.2 gives

PROPOSITION 2.3. *$F_{\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)}$ is multiply mixing with respect to Lebesgue measure.*

REMARK 2.4. Simple Fourier analysis also establishes this conclusion, but while linearity is helpful for the Hopf argument, it is indispensable for Fourier analysis.

REMARK 2.5. Instead of ergodicity of an irrational rotation, one can use Theorem 5.1, and it may be of interest to read the proofs with $\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$ in mind.

REMARK 2.6. A volume-preserving C^1 perturbation of $F_{\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)}$ is a topologically conjugate Anosov diffeomorphism for which the local product charts can be chosen to be differentiable. (More generally, any volume-preserving Anosov diffeomorphism of $\mathbb{T}^2$ is topologically conjugate to a hyperbolic automorphism, and the local product charts can be chosen to be differentiable.) Thus, we have a local product structure and absolute continuity for free and obtain multiple mixing from Theorem 5.1 and Theorem 2.2.

REMARK 2.7. In contrast with Lebesgue measure, the measure that assigns $1/4$ to each of the points $\pm 1/5\binom{1}{2}$ and $\pm 1/5\binom{2}{4}$ has 2 ergodic components. The reader is encouraged to check where this affects our proofs.

The contracting lines of the partially hyperbolic automorphism in Example 1.14 are generated by a vector whose components are rationally independent, hence project to the orbits of an ergodic flow [18, p. 147]. Theorem 2.2 gives:

PROPOSITION 2.8. *F_W in Example 1.14 is multiply mixing.*

3. THE TWO-SIDED HOPF ARGUMENT YIELDS MIXING

The assumption in Theorem 2.2 that W^{ss} is ergodic is rather strong, and the classical Hopf argument is based on joint ergodicity of W^{ss} and W^{su}. To improve this to mixing, we need to augment the conclusion of Proposition 2.1 to include W^{su}-saturation as well. This requires a slightly subtle argument.

THEOREM 3.1 ([13, Theorem 3])**.** *If (X,μ) is a metric Borel probability space, $f\colon X\to X$ invertible μ-preserving and $\varphi\in L^2(\mu)$, then any weak accumulation point of $U_f^n(\varphi)$ as $n\to+\infty$ is W^{ss}- and W^{su}-saturated.*

Proof ([7, 13]). Denote by $I\subset L^2(\mu)$ the (closed) subspace of functions subordinate to W^{ss} and W^{su} and by $I^\perp := \{\varphi\in L^2 \mid \langle\varphi,\psi\rangle = 0 \text{ for } \psi\in I\}$ its orthocomplement. To show $U_f^{n_i}(\varphi)\xrightarrow[i\to\infty]{\text{weakly}}\psi\Rightarrow\psi\in I$ take $\varphi=\varphi_I+\varphi^\perp\in I\oplus I^\perp=L^2$ and a subsequence with $U_f^{n_{i_k}}(\varphi_I)\xrightarrow[k\to\infty]{\text{weakly}}\psi_I\in I$ and $U_f^{n_{i_k}}(\varphi^\perp)\xrightarrow[k\to\infty]{\text{weakly}}\psi^\perp\perp I$. Then $\psi=\psi_I+\psi^\perp$, and we are done if we find a ψ' with $\langle\psi^\perp,\psi^\perp\rangle=\langle\varphi^\perp,\psi'\rangle=0$.

By Proposition 2.1, $\psi^\perp$ is subordinate to W^{ss}, and hence so is any $U_f^{-n}(\psi^\perp)$ and any weak limit $\psi'=\lim_{i\to\infty}U_f^{-n_i}(\psi^\perp)$, while Proposition 2.1 applied to $\psi^\perp$ and f^{-1} implies that ψ' is subordinate to W^{su} as well, i.e., $\psi'\in I$. Thus

$$0=\langle\varphi^\perp,\psi'\rangle=\lim_{i\to\infty}\langle\varphi^\perp,U_f^{-n_i}(\psi^\perp)\rangle=\lim_{i\to\infty}\langle U_f^{n_i}(\varphi^\perp),\psi^\perp\rangle=\langle\psi^\perp,\psi^\perp\rangle. \qquad \square$$

Theorem 3.1 can alternatively be obtained from the following result:

THEOREM 3.2 (Derriennic–Downarowicz [15, Théorème 2.4])**.** *A weak accumulation point of $(U_f^n(\varphi))_{n\in\mathbb{N}}$ is a weak accumulation point of $(U_f^{-n}(\phi))_{n\in\mathbb{N}}$ for some ϕ.*

Theorem 3.1 has the following consequences, as noted in [13]:

THEOREM 3.3. *If (X,μ) is a metric Borel probability space, $f\colon X\to X$ invertible μ-preserving, W^{ss}, W^{su}* jointly ergodic*, then f is mixing.*

THEOREM 3.4. *If (X,μ) is a metric Borel probability space, $f\colon X\to X$ invertible μ-preserving, and*

$$\varphi\in L^2(\mu)\ f\text{-invariant, } W^{ss}\text{-saturated and } W^{su}\text{-saturated} \Rightarrow \varphi \overset{\text{a.e.}}{=} \text{const.},$$

then f is ergodic.

Proof. An f-invariant φ is a weak accumulation point of $\varphi=\prod_{i=1}^{N}\varphi_i\circ f^{n_i}$, hence W^{ss}- and W^{su}-saturated by Proposition 2.1, hence constant by assumption. ☐

Theorem 3.1 also holds for flows (mutatis mutandis), and thus we get the following corollary:

COROLLARY 3.5. *Let X be a metric space, $f^t\colon X\to X$ a flow, μ an f^t-invariant Borel probability measure. If*

$$\varphi\in L^2(\mu)\ f^t\text{-invariant, } W^{ss}\text{-saturated and } W^{su}\text{-saturated} \Rightarrow \varphi \overset{\text{a.e.}}{=} \text{const.},$$

then f^t is ergodic, and joint ergodicity of W^{ss}, W^{su} implies that f^t is mixing.

Our aim is to obtain multiple mixing easily, but Theorem 3.3 is interesting because of its weak hypotheses. It applies where other methods do not [7].

4. ABSOLUTE CONTINUITY AND PRODUCT SETS

We now apply these results to the hyperbolic toral automorphisms of Example 1.13 to demonstrate the classical use of the Hopf argument to get ergodicity, except that Theorem 3.3 yields mixing instead.

PROPOSITION 4.1. *If $A\in GL(m,\mathbb{Z})$ is hyperbolic, then the induced automorphism F_A of $\mathbb{T}^m$ is mixing with respect to Lebesgue measure.*

Proof. For $q\in\mathbb{T}^m$ the *stable* and *unstable subspaces* at q in (1) are

$$W^{ss}(q)=\pi(E^-+q) \text{ and } W^{su}(q)=\pi(E^++q),$$

where $E^\pm$ are the contracting and expanding subspaces of A and $\pi\colon\mathbb{R}^m\to\mathbb{T}^m$ is the projection. Suppose $\varphi\in L^2$ is W^{ss}- and W^{su}-saturated, i.e., there is a conull $G\subset\mathbb{T}^n$ such that $x,y\in G,\ y\in W^{ss}(x)\Rightarrow\varphi(x)=\varphi(y)$ and $x,y\in G,\ y\in W^{su}(x)\Rightarrow\varphi(x)=\varphi(y)$. We will prove that $\varphi\overset{\text{a.e.}}{=}$ const., and Theorem 3.3 then implies mixing.

Let $D^\pm\subset E^\pm$ be small disks and $q\in\mathbb{T}^m$. Then q has a neighborhood that is up to rotation and translation of the form $D^-\times D^+$, and $C:=G\cap(D^-\times D^+)$ has full Lebesgue measure in $D^-\times D^+$, i.e., if $\mu^\pm$ denotes the normalized Lebesgue measure on $D^\pm$ and $\mu=\mu^-\times\mu^+$, then $\int_{D^-\times D^+}\chi_C\,d\mu=1$. By the Fubini Theorem

$$1=\int_{D^-\times D^+}\chi_C\,d\mu=\int_{D^-}\int_{D^+}\chi_C\,d\mu^+\,d\mu^-, \text{ so } \int_{D^+}\chi_C(u,\cdot)\,d\mu^+=1 \text{ for } \mu^-\text{-a.e. } u\in D^-.$$

Fix such a $u_0 \in D^-$, and note that by construction $C^- := D^- \times \big(C \cap (\{u_0\} \times D^+)\big)$ has full Lebesgue measure.[2] If $(u,v),(u',v') \in C^- \cap C$, a set of full measure, then

$$\varphi(u,v) = \varphi(u_0,v) = \varphi(u_0,v') = \varphi(u',v').$$

This applies to any such neighborhood of an arbitrary $q \in \mathbb{T}^n$, so $\varphi \overset{\text{a.e.}}{=} \text{const}$. □

This is how Hopf established the ergodicity of geodesic flows of manifolds of negative curvature. The method was extended to geodesic flows of higher-dimensional manifolds by Anosov. The pertinent discrete-time counterpart are Anosov diffeomorphisms, which include the F_A above. As the preceding argument shows, higher-dimensionality does not directly affect the intrinsic difficulty of the argument. The barrier that Hopf faced and Anosov overcame is related to the use of the Fubini Theorem above—except in Hopf's context, where local product neighborhoods are indeed diffeomorphic to euclidean patches, one needs to establish the *absolute continuity* of the invariant foliations on each such patch to apply the Fubini Theorem (see, e.g., [6, Chapter 6]). This is a natural point at which to define center-stable and -unstable sets

$$W^{cs}(x) := \{y \in X \mid \{d(f^n(x), f^n(y))\}_{n\in\mathbb{N}} \text{ is bounded}\},$$
$$W^{cu}(x) := \{y \in X \mid \{d(f^{-n}(x), f^{-n}(y))\}_{n\in\mathbb{N}} \text{ is bounded}\}.$$

DEFINITION 4.2. Let (X,μ) be a metric Borel probability space, $f\colon X \to X$ invertible μ-preserving, $i \in \{ss, cs\}$, $j \in \{su, cu\}$. We say that $V \subset X$ is an (i,j)-*product set* if for $x \in V$ and $k \in \{i,j\}$ there are $W^k_{\text{loc}}(x) \subset W^k(x)$ and a measurable map $[\cdot,\cdot]\colon V \times V \to X$ with $[x,y] \in W^i_{\text{loc}}(x) \cap W^j_{\text{loc}}(y)$.

We say that W^i is *absolutely continuous* on an (i,j)-*product set* V (with respect to μ) if for each $x \in V$ and $k \in \{i,j\}$ there are measures μ^k_x on $W^k_{\text{loc}}(x)$ with $\mu^j_x(N) = 0 \Rightarrow \mu^j_y([N,y]) = 0$ and $\phi \in L^1(\mu) \Rightarrow \int_V \phi\, d\mu = \int_{W^i_{\text{loc}}(z)} \int_{W^j_{\text{loc}}(x)} \phi\, d\mu^j_x\, d\mu^i_z(x)$.

Then one obtains [6, Chapter 6]:

PROPOSITION 4.3. *Volume-preserving Anosov diffeomorphisms are mixing.*

Theorem 3.3 can be applied well beyond this completely hyperbolic case. With the terminology of [9] we have

THEOREM 4.4. *Let f be C^2, volume-preserving, partially hyperbolic, and center bunched. If f is essentially accessible, then f is mixing.*

Proof. Every bi-essentially saturated set is essentially bisaturated [9, Corollary 5.2], so Theorem 3.3 applies by essential accessibility [9, p. 472]. □

The main result of Burns and Wilkinson [9, Theorem 0.1] is that f is ergodic and in fact has the Kolmogorov property. They obtain ergodicity from [9, Corollary 5.2] by the Hopf argument, so by Theorem 3.3 one obtains mixing directly. The Kolmogorov property is then obtained by invoking a result of Brin and Pesin [5] that the Pinsker algebra is bi-essentially saturated in this context.

[2] One might at this time revisit Remark 2.7.

Our point is that here, too, the Hopf argument alone provides mixing rather than just ergodicity without any "high-tech" ingredients.

5. APPLICATIONS: MULTIPLE MIXING

Theorem 3.3 says that f is mixing if $\varphi \in L^2(\mu)$, W^{ss}- and W^{su}-saturated $\Rightarrow$ $\varphi \overset{\text{a.e.}}{=}$ const., and in the previous section we established the "if" part of the statement. Likewise, Theorem 2.2 says that if every W^{ss}-saturated $\varphi \in L^2$ is constant a.e., then f is multiply mixing, and we now (on page 13) verify this "if" statement—in remarkable generality, such as in the original context (of uniformly hyperbolic dynamical systems) in which the Hopf argument applies in the manner shown in Section 3. The result does not use the contraction on W^{ss}; thus it also applies to W^{cs}, such as in Example 1.14 or to the *weak*-stable foliation of a flow.

THEOREM 5.1. *If (X,μ) is a metric Borel probability space, $f\colon X \to X$ invertible μ-preserving ergodic, $i \in \{ss, cs\}$. If W^i is absolutely continuous on an (i, su)-product set V, and $\mu(f^{-1}(V) \cap V) > 0$, then W^i is ergodic.*

COROLLARY 5.2. *If (X,μ) is a metric Borel probability space, $f\colon X \to X$ invertible μ-preserving totally ergodic, W^{ss} absolutely continuous on an (ss, su)-product set V with $\mu(V) > 0$. Then f is multiply mixing.*

Proof. The Poincaré Recurrence Theorem gives an $N \in \mathbb{N}$ with $\mu(f^{-N}(V) \cap V) > 0$. Apply Theorem 5.1 to f^N, then Theorem 2.2 to f. □

Theorem 1.8 makes it easy to establish total ergodicity. For instance:

THEOREM 5.3. *Let X be a separable metric space, μ a Borel probability measure with connected support, $f\colon X \to X$ an invertible μ-preserving transformation. If W^{ss} is absolutely continuous on open (ss, su)-product sets that cover the support of μ, then f is totally ergodic and thus multiply mixing by Corollary 5.2.*

Proof. Apply Theorem 1.8: An f-invariant function is W^{ss}- and W^{su}-saturated, hence by absolute continuity a.e. constant on these product sets. A function on a connected set is a.e. constant if it is a.e. locally constant. □

REMARK 5.4. This applies to volume-preserving Anosov diffeomorphisms [6, Chapter 6] but we do not use exponential behavior, differentiability or compactness.

THEOREM 5.5. *The Liouville measure for dispersing billiards (Example 1.15) and for polygonal billiards with pockets (Example 1.16) is multiply mixing.*

Proof. For dispersing billiards, Sinai's Fundamental Theorem of the theory of dispersing billiards [10, Theorem 5.70] provides product sets [10, Proposition 7.81] with absolutely continuous holonomies [10, Theorem 5.42], which implies the absolute continuity property we use. Theorem 3.3 then establishes mixing and hence total ergodicity, which by Corollary 5.2 implies mixing of all orders. This also works for polygonal billiards with pockets [11, Theorem 4.1]. □

THEOREM 5.6. *The Katok map (Example 1.17) is multiply mixing.*

Proof. It is totally ergodic and the stable and unstable partitions are homeomorphic to those of $F_{\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)}$ (Example 1.17), so there is a product neighborhood, which hence has positive measure. Absolute continuity on this neighborhood follows from Pesin theory, so we can apply Corollary 5.2. □

In fact, the proofs of Theorem 5.1 and Corollary 5.2 applied to the pieces of the ergodic decomposition of μ yield:

COROLLARY 5.7. *Let X be a metric space, μ a Borel probability measure, $f\colon X\to X$ μ-preserving invertible, $i\in\{ss,cs\}$ such that every point is in an (i,su)-product set where W^i is absolutely continuous. If $\varphi\colon X\to\mathbb{R}$ is W^i-saturated, then there is a measurable f-invariant $n\colon X\to\mathbb{N}$ with $\varphi(f^{n(x)}(x))=\varphi(x)$ a.e.*

LEMMA 5.8. *Absolute continuity of W^i on $V_f:=f^{-1}(V)\cap V$ implies absolute continuity of $T\colon V_f\to X$, $x\mapsto T(x):=[f(x),x]$, i.e., $T_*\mu\ll\mu$.*

Proof. If $N\subset V_f$ and $\mu(N)=0$, then there is a W^{su}_{loc}-saturated null set N_W such that for $z\notin N_W$ we have $\int_{W^{su}(z)}\chi_N\,d\mu^{su}_z=0$ as well as, by f-invariance of μ and absolute continuity, $\int_{W^{su}(z)}\chi_{T(N)}\,d\mu^{su}_z=0$. Then

$$\begin{aligned}\int\chi_{T(N)}\,d\mu&=\int_{W^i_{\mathrm{loc}}(z)\smallsetminus N_W}\int_{W^{su}_{\mathrm{loc}}(x)}\chi_{T(N)}\,d\mu^{su}_x\,d\mu^i_z(x)+\int_{N_W}\chi_{T(N)}\,d\mu\\&=\int_{W^i_{\mathrm{loc}}(z)\smallsetminus N_W}0\,d\mu^i_z(x)+0.\end{aligned}$$
□

We adapt an idea of Thouvenot [22, Theorem 1], [14, Exercice 7, p. 50], [15, Proposition 1.2]: $d(f^{-n}(x),f^{-n}(T(x)))\to0$ pointwise on $V_f\cap T^{-1}V_f$, hence by the Egorov theorem uniformly on some $U\subset V_f\cap T^{-1}V_f$ with $\mu(U)>0$. Then $T_n:=\begin{cases}f^{-n}\circ T\circ f^n&\text{on } f^{-n}(U)\\ \mathrm{Id}&\text{elsewhere}\end{cases}\xrightarrow[n\to\infty]{\text{pointwise}}\mathrm{Id}$, and T_n has Radon–Nikodym derivative $g_n:=\left[\frac{dT_{n*}\mu}{d\mu}\right]=\left[\frac{dT_*\mu}{d\mu}\right]\circ f^n$ on $f^{-n}(U)$ (and 1 elsewhere); this is uniformly integrable, i.e., $\sup_{n\in\mathbb{N}}\int_{\{g_n>M\}}g_n\,d\mu\xrightarrow[M\to\infty]{}0$.

LEMMA 5.9. *Let X be a metric space with probability measure μ, $T_n\colon X\to X$ such that $T_n\to\mathrm{Id}$ a.e., $T_{n*}\mu\ll\mu$, and $g_n:=\left[\frac{dT_{n*}\mu}{d\mu}\right]$ is uniformly integrable. Then $\|\varphi\circ T_n-\varphi\|_1\xrightarrow[n\to\infty]{}0$ for all $\varphi\in L^\infty$. ($\|\cdot\|_p$ denotes the L^p-norm.)*

Proof. If ψ is continuous with $\|\psi\|_\infty\le\|\varphi\|_\infty$, then

$$\|\varphi\circ T_n-\varphi\|_1\le\|(\varphi-\psi)\circ T_n\|_1+\|\psi\circ T_n-\psi\|_1+\|\psi-\varphi\|_1,$$

and $\|\psi\circ T_n-\psi\|_1\xrightarrow[n\to\infty]{}0$ by the Bounded Convergence Theorem. For $\epsilon>0$, uniform integrability provides an M such that the last summand in

$$\|(\varphi-\psi)\circ T_n\|_1=\int|\psi-\varphi|_1 g_n\,d\mu\le M\|\psi-\varphi\|_1+2\|\varphi\|_\infty\int_{\{g_n>M\}}g_n\,d\mu$$

is less than $\epsilon/2$. Choose ψ such that $\|\psi-\varphi\|_1<\frac{\epsilon/2}{M+1}$. □

Proof of Theorem 5.1. Let $\varphi \in L^\infty$ be W^i-saturated. We show φ is f-invariant. If $\varepsilon > 0$, then $T_n(x) \in W^i(f(x))$ for all $x \in f^{-n}(U)$ implies that

$$(4)\quad \mu\Big(f^{-n}(U) \cap \big\{|\varphi \circ f - \varphi| > \varepsilon\big\}\Big) = \mu\Big(f^{-n}(U) \cap \big\{|\varphi \circ T_n - \varphi| > \varepsilon\big\}\Big) \xrightarrow[n\to\infty]{\text{(Lemma 5.9)}} 0.$$

With $B := \big\{|\varphi \circ f - \varphi| > \varepsilon\big\}$, the Mean Ergodic Theorem and ergodicity of f imply

$$\frac{1}{n}\sum_{k=0}^{n-1} \chi_U \circ f^k \xrightarrow[n\to\infty]{L^2} \mu(U), \quad \text{hence} \quad \frac{1}{n}\sum_{k=0}^{n-1} \chi_U \circ f^k \chi_B \xrightarrow[n\to\infty]{L^2} \mu(U)\chi_B,$$

so $0 \xleftarrow[n\to\infty]{(4)} \frac{1}{n}\sum_{k=0}^{n-1} \mu\Big(f^{-n}(U) \cap \big\{|\varphi \circ T_n - \varphi| > \varepsilon\big\}\Big) \xrightarrow[n\to\infty]{} \mu(U)\,\mu(B)$. Since $\mu(U) > 0$, we have $\mu(B) = 0$. ϵ was arbitrary, so φ is f-invariant, hence constant a.e. □

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